ON THE STABILITY OF REGULAR PRECESSION OF A SATELLITE

(OB USTOICHIVOSTI REGULIARNOI PRETSESSII SPUTNIKA)

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Considered is the motion of a rigid body (satellite) in the field of attraction of a fixed center. Sufficient conditions of stability are derived for the regular precession of a satellite which are compared with the necessary conditions.

Within the accuracy of terms of order $(l/R)^2$ (l is the characteristic linear dimension of the satellite and **R** is the radius vector from the center of attraction to its center of mass) the motion of the satellite center of mass may be considered independent of its relative motion and is consequently governed by Kepler's laws. Let the satellite possess dynamic symmetry with A and C being its principal central moments of inertia (equatorial and axial respectively). The moment of the gravitational forces about the satellite center of mass within the accuracy of terms of order l/R is equal to [1]

$$\mathbf{L} = 3\mu R^{-5}(C-A)(\mathbf{Rz})(\mathbf{R}\times\mathbf{z}) \qquad (1)$$

Here μ is the gravitational constant and $\mathbf{z}$ is the unit vector along the dynamic symmetry of the satellite. We will find the conditions for which the satellite can perform regular precession. The moment of external forces must in this case be equal to

$$\mathbf{L} = (\boldsymbol{\omega}_1\times\mathbf{z})\,[C\Omega + (C-A)\,\omega_1\cos\theta] \qquad (2)$$

where $\boldsymbol{\omega}_1$ is the constant angular velocity vector of the precession, Ω is the angular velocity of the body and θ the nutation angle. Comparing (1) and (2) we note that if $L\neq 0$ the vectors $\boldsymbol{\omega}_1$, **R** and **z** are coplanar. It follows from the kinematic properties of Kepler motion and regular precession that this is possible only in a circular orbit, whereby $\boldsymbol{\omega}_1=\boldsymbol{\omega}_0$ ($\boldsymbol{\omega}_0$ is the angular velocity vector for orbital motion, $\omega_0^2=\mu R^{-3}$). If, on the other hand, $L=0$ then the vectors **R** and **z** are at all times either collinear or orthogonal (the trivial case $A=C$ is not considered). In the presence of regular precession, this is also possible only in a circular orbit and for $\boldsymbol{\omega}_1=\boldsymbol{\omega}_0$. Therefore, in the following the orbit will be considered circular.

Let α, β and γ be the direction cosines of the unit vector **z** relative to the satellite mass center velocity vector, the normal to the orbit plane ($\boldsymbol{\omega}_0$) and the radius vector **R**, respectively. Obviously,

$$\alpha^2+\beta^2+\gamma^2=1 \qquad (3)$$

In addition, $\beta = \cos\theta$ and with no loss of generality it is assumed that $\beta \geqslant 0$ (in the opposite case the direction of **z** is reversed). Equating (1) and (2) with $\omega_1 = \omega_0$, we find the possible regimes of regular precession which can be conveniently represented in the form of three one-parameter families

$$\theta = 0, \qquad \alpha = \gamma = 0, \qquad \beta = 1, \qquad \Omega = \Omega_0 \tag{4}$$

$$\theta = \theta_0 \neq 0, \qquad \gamma = 0, \qquad \beta = \cos\theta_0, \qquad \Omega = (A - C)\, C^{-1}\omega_0 \cos\theta_0 \tag{5}$$

$$\theta = \theta_0 \neq 0, \qquad \alpha = 0, \qquad \beta = \cos\theta_0, \qquad \Omega = 4\,(A - C)\, C^{-1}\omega_0 \cos\theta_0 \tag{6}$$

The parameter in (4) is Ω_0, and in (5) and (6) it is the angle θ_0 located, by assumption $(\beta \geqslant 0)$ in the first quarter. The case (4) is that of uniform rotation in the orbit plane. In the case (5) the satellite axis **z** is perpendicular to the radius vector **R**, while in (6) it is perpendicular to the mass center velocity vector, where moreover $L = 0$ in the cases of (4) and (5). Solutions (4) to (6) have been obtained by a different method in the works of V.T. Kondudar' and Duboshin (See, for example, [2]).

Let us find the sufficient stability conditions for the motions (4) to (6) by the method of N.G. Chetaev. The first integral of the equations of the satellite relative motion for a circular orbit [1] and for the case of dynamic symmetry is of the form

$$A\,(p^2 + q^2) + Cr^2 + 3\omega_0^2\,(C - A)\,\gamma^2 + \omega_0^2\,(A - C)\,\beta^2 = h$$

Here r is the projection of the relative angular velocity $\boldsymbol{\omega} - \boldsymbol{\omega}_0$ of the satellite on the **z**-axis ($\boldsymbol{\omega}$ is the absolute angular velocity), while p and q are its projections in any two mutually perpendicular directions in the equatorial plane of the satellite. Also, because of the condition of dynamic symmetry, we have the first integral $r + \omega_0\beta = r_0$. We will select the constants k_1 and k_2 such that the first integral $V = h + k_1 r_0 + k_2 r_0^2$ obtain a strict minimum for the values of its arguments corresponding to one of the motions in (4) to (6). Note that for all these motions $p = q = 0$ and $r = \Omega$.

Let us first consider the undisturbed motion (4) and let for the disturbed motion $p = u_1$, $q = u_2$, $r = \Omega_0 + u_3$, $\alpha = u_4$ and $\gamma = u_5$, while β is eliminated by means of (3). Then, as can be easily seen, the first integral $V_1 = h - 2C\Omega_0 r_0$ considered as a function of the variables u_i, has a strict minimum at the point $u_i = 0$ ($i = 1,2,3,4,5$), if simultaneously

$$C\Omega_0 + \omega_0\,(C - A) > 0, \qquad C\Omega_0 + 4\omega_0\,(C - A) > 0$$

Hence, on the strength of the known theorem for the stability of motion, it follows that for the stability of motion (4) it is sufficient that

$$\Omega_0 > (A - C)\,\omega_0 / C \quad \text{for } A \leqslant C \tag{7}$$

$$\Omega_0 > 4\,(A - C)\,\omega_0 / C \quad \text{for } A \geqslant C$$

Let us consider now the motion (5) as undisturbed and let

$$p = u_1, \qquad q = u_2, \qquad r = (AC^{-1} - 1)\,\omega_0 \cos\theta_0 + u_3$$

$$\beta = \cos\theta_0 + u_4, \qquad \gamma = u_5$$

while α is eliminated by means of (3). In this case, the function

$$V_2 = h - 2A\omega_0 r_0 \cos\theta_0 + C^2A^{-1}r_0^2$$

will be, up to the accuracy of the constant term, a positive-definite quadratic form in u_i for the condition $A < C$ which will be sufficient for the stability of motion (5).

Finally, in order to investigate the stability of motion (6) we let

$$p = u_1, \quad q = u_2, \quad r = 4\,(AC^{-1} - 1)\,\omega_0 \cos\theta_0 + u_3$$
$$\alpha = u_4, \ \beta = \cos\theta_0 + u_5$$

and eliminate γ by means of (3). The first integral

$$V_3 = h - 8\,(A - C)\,\omega_0 r_0 \cos\theta_0$$

is, up to the accuracy of the constant term, a positive-definite quadratic form in u_i for $A > C$. This condition is sufficient for the stability of motion (6).

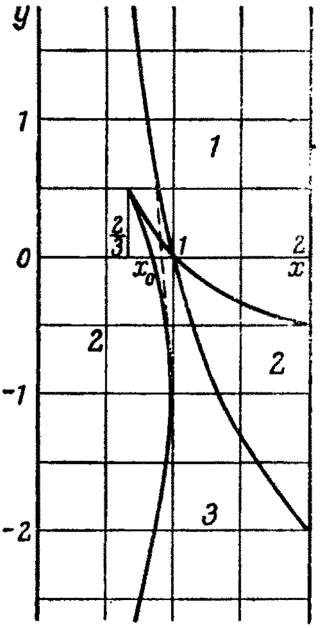

Fig. 1

Let us compare the derived sufficient conditions with the necessary conditions for stability of the motions (4) to (6). These conditions are obtained if the equations of motion are linearized near the solutions (4) to (6) and require that the real parts of all roots of the characteristic equation for the system of linear approximation be nonpositive.

The investigation of stability of motion (4) leads to the necessary conditions

$$a \geqslant 2\sqrt{b}, \qquad b \geqslant 0 \tag{8}$$

where

$$a = (xy + x - 2)^2 + (xy + x - 1) + (xy + 4x - 4)$$
$$b = (xy + x - 1)\,(xy + 4x - 4), \qquad x = C\,/\,A,$$
$$y = \Omega_0\,/\,\omega_0 \tag{9}$$

Note that $x \leqslant 2$ for any dynamically symmetrical rigid body. Conditions (8) and (9) have been derived in a somewhat different notation by Tomson [3]. Substituting (9) into (8) these conditions are simplified. For $x \geqslant 1$ it is necessary for stability that one of the following two inequalities be satisfied

$$y \geqslant x^{-1} - 1, \qquad y \leqslant 4\,(x^{-1} - 1) \tag{10}$$

For $x \leqslant 1$ it is necessary that either

$$y \geqslant 4\,(x^{-1} - 1) \tag{11}$$

or that the following two conditions be fulfilled simultaneously

$$y \leqslant x^{-1} - 1, \qquad \sqrt{1 - x - xy} + \sqrt{4 - 4x - xy} \leqslant 2 - x - xy \tag{12}$$

The sufficient conditions (7) differ only by the strict sign of inequality from the first condition in (10) and the condition in (11). Fig. 1 shows the regions defined by the inequalities (10) to (12) on the surface of the parameters x, y. The region 1 is truly stable (sufficient conditions are fulfilled), the regions 2 are unstable (necessary conditions are violated), while in the region 3 only the necessary conditions are satisfied. The boundaries of the regions 1,2 and 2,3 have asymptote $x = 0$, while the boundary of the regions 2,3 for $x < 1$ intersects the x-axis at $x = x_0 = (3\sqrt{5} - 5)\,/\,2 \approx 0.854$. Note that in the region 3 there exist points ($y \to -\infty$, as well as the line $x = 1$) corresponding to the known stable regimes of motion.

In [3] the conditions of the type (10) to (12) are missing, and the presented diagram of the stability regions is incorrect, especially near $x = 1$ (the dashed curve in Fig. 1). It has no region of instability for $x > 1$.

The necessary conditions of stability for motions (5) and (6) have been derived in the work of Duboshin [2]. For the stability of motion (5) it is necessary that $A \leqslant C$. The above derived sufficient condition $A < C$ differs from the necessary one only by the strict inequality sign.

For the stability of motion (6) it is necessary that either $A \geqslant C$ $(x \leqslant 1)$, or that following two conditions be simultaneously fulfilled [2]

$$x \geqslant \frac{4}{3}, \qquad \cos^2\theta_0 \geqslant \frac{18x^2 - 27x + 8 + 2(3x-2)\sqrt{(3x-1)(3x-4)}}{27x^2(x-1)} \tag{13}$$

where x is defined by Formula (9). The regions of true stability 1 and the instability 2, and the region 3 where the necessary conditions (13) are fulfilled, are shown in Fig. 2 for the motion (6).

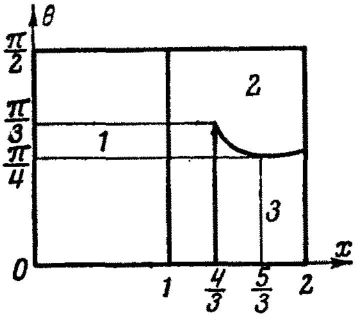

Fig. 2

We will consider some particular cases. For $\Omega_0 = -\omega_0$ $(y = -1)$ solution (4) describes the translational motion of the satellite, the axis **z** of dynamic symmetry of which is perpendicular to the orbit plane (Ω_0 is the relative angular velocity). It follows from the above-obtained conditions that this motion can be stable only for a satellite where $1 \leqslant C/A \leqslant 4/3$.

For $\Omega_0 = 0$ $(y = 0)$ the motion (4) represents the position of relative equilibrium of the satellite in a circular orbit, for which satellite's **z**-axis is perpendicular to the orbit plane. Conditions (7) and (12) show that this position of equilibrium is stable for $A < C$ and unstable for $C/A < x_0 \approx 0.854$.

For $\theta_0 = \pi/2$ the motions (5) and (6) pass into two different positions of relative equilibrium of the satellite in a circular orbit: the **z**-axis is directed along the tangent to the orbit or along the radius vector **R**. The derived conditions show that the orientation of the satellite axis **z** along the tangent to the orbit is stable for $A < C$ and unstable for $A > C$, while its orientation along the radius vector is stable for $A > C$ and unstable for $A < C$.

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